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ON THE DETERMINATION OF THE SOUND FIELD IN A LAYER

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We consider a method for determining the sound field in a two-dimensional layer. The method we present combines the usual method of reflected plane waves with a summation from graphs. It makes it comparatively easy to take into account the complex interference pattern due to the transformation of the various waves at the boundaries of the layer and to obtain integral relations for the sound potentials. When the layer thickness tends to infinity, the problem reduces to one concerning the reflection of sound waves at the interface of two media. We study the potentials of normal waves in the case of a harmonic source in a solid.

1. Fundamental relations. The field of a cylindrically symmetric point radiator with its axis perpendicular to the plane of the layer (Fig. 1) can be described by the potential ψ of the shear waves, polarized in the plane of incidence, and by the potential φ of the compression waves, which satisfy corresponding wave equations [1-3]. Here the rotating shear stresses applied to the lateral surface of the cylinder are not taken into account. These potentials are expanded in terms of plane waves and, along with a Fourier integral expansion of their temporal parts, and they have the form

$$\varphi = \frac{1}{\pi} \operatorname{Re} \int_0^\infty e^{ikct} dk \int_0^{\pi/2 - i\infty} \int_0^{2\pi} f_0(k, \vartheta) \exp\{ik[x \sin\vartheta \cos\varphi + y \sin\vartheta \sin\varphi \pm (z - z_0)\cos\vartheta]\} \sin\vartheta d\vartheta\, d\varphi \tag{1.1}$$

(the analogous expression for ψ contains the function $g_0(k\vartheta)$ instead of the function $f_0(k\vartheta)$ and the z-projection of the wave vector $\varkappa \cos\gamma$ instead of $k\cos\vartheta$ in the second term of the exponent). Here c is the speed of a compression wave in the solid and k

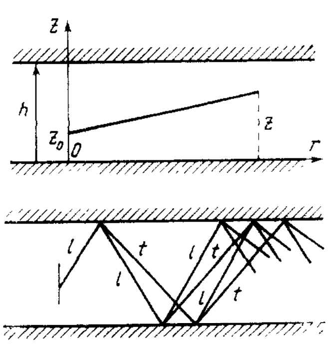

Fig. 1

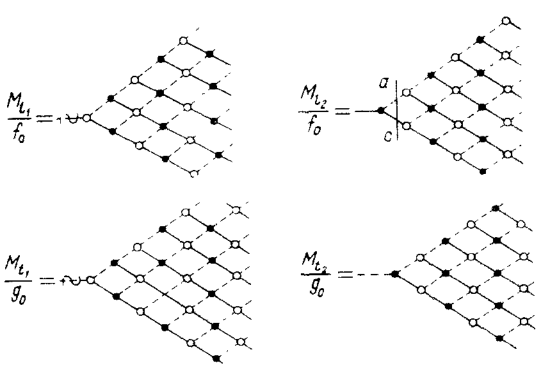

Fig. 2

and $\varkappa$ are, respectively, the wave vectors of longitudinal and shear waves. The angles ϑ and γ are connected by the relation $k \sin \vartheta = \varkappa \sin \gamma$. The functions $f_0(k, \vartheta)$ and $g_0(k, \vartheta)$ are determined from the conditions of equilibrium on the boundary of the radiator and the medium [1, 4]. The plus sign in the exponent is chosen when $z - z_0 > 0$ and the minus sign when $z - z_0 < 0$. For definiteness we assume that the first of these inequalities is satisfied.

2. Method of graphical wave summation. In [4] the process of plane wave propagation in a solid was represented graphically by the scheme shown in Fig. 2. In what follows, we propose to associate to each graph of this scheme real analytic expressions and to consider these as the sums of all possible plane waves obtained when the incident wave undergoes multiple reflections from the boundaries of the layer. In fact, by using the ray pattern of the multiple reflection of plane waves in an elastic layer, we can analyze the propagation of these waves graphically. In this analysis it is necessary to associate to each ray of the longitudinal or shear waves a continuous or a dotted line, respectively, and to regard it as an edge of the graph; points of ray reversal then become the boundary points of an edge or the vertices of the graph, being indicated by dark circles on the lower boundary and by open circles on the upper boundary. Then each configuration of the scheme is a graph. Putting each edge of the graph into correspondence with the analytical expression for the potential of the corresponding plane wave, and making each vertex correspond to the coefficient of reflection of this wave, then, to the graph in its entirety there will be associated the analytical expression for the total potential of the plane waves in the elastic layer.

There exist four graphs such that the analytical expressions

$$M_{l1},\ M_{l2},\ M_{t1},\ M_{t2}$$
$$(M_{li}^{\circ} = M_{li} / f_0,\ M_{ti}^{\circ} = M_{ti} / g_0,\ i = 1, 2)$$

associated with them represent the sum of all the plane waves propagating in the layer, with angles of incidence ϑ for the longitudinal waves and γ for the shear waves. Here, and in what follows, quantities with the subscripts l and t, respectively, will refer to longitudinal and transverse waves.

Cutting an arbitrary graph at a vertex, we obtain three parts, one of which is the incident wave written in explicit form, the other two being graphs of the same type as the original one. What this means, in fact, is that the graphs thus obtained anew represent the sum of all the plane waves if in them we consider as the "incident" wave the first reflected wave. A similar operation with the remaining graphs enables us to mutually interrelate them and thus to obtain a closed system of equations for them.

A section of the graph M_{l2}° along the line ac (Fig. 2) yields, to the right, a sum of graphs of the type M_{l1}° and M_{t1}°

$$M_{l2}^{\circ} = F_l + V_{ll}^{(2)*} M_{l1}^{\circ} + V_{lt}^{(2)*} M_{t1}^{\circ} \tag{2.1}$$

$$F_l = \exp\left[b_l (z - z_0)\right] + V_{ll}^{(2)} \exp\left[b_l (2h - z - z_0)\right] + V_{lt}^{(2)} \exp\left[b_l (h - z_0) + b_t (h - z)\right] \tag{2.2}$$

$$V_{ll}^{(2)*} = V_{ll}^{(2)} \exp\left[2b_l (h - z_0)\right], \qquad b_l = ik \cos \vartheta$$
$$V_{lt}^{(2)*} = V_{lt}^{(2)} \exp\left[(b_l + b_t)(h - z_0)\right], \qquad b_t = i\varkappa \cos \gamma$$

Here $V_{ll}^{(i)}$, $V_{lt}^{(i)}$, $V_{tl}^{(i)}$ and $V_{tt}^{(i)}$ are the reflection coefficients of the plane waves; the superscripts $i=1$ and 2 refer to waves reflected from the lower and upper boundaries of the layer, respectively [1, 3]. The terms on the right side of Eq. (2.2) correspond to the incident wave and the two reflected waves.

The graph M_{l1}° yields the following equation:

$$M_{l1}^{\circ} = V_{ll}^{(1)*} M_{l2}^{\circ} + V_{lt}^{(1)*} M_{t2}^{\circ}$$
$$V_{ll}^{(1)*} = V_{ll}^{(1)} \exp(2b_l z_0), \qquad V_{lt}^{(1)*} = V_{lt}^{(1)} \exp[(b_l + b_t) z_0]$$

The absence of terms analogous to F_l in Eq. (2.1) stipulates that the incident wave is nonreal (line with a tilde in Fig. 2) and that the reflected waves have already been taken into account in M_{l2}° and M_{t2}° as incident waves.

3. General solution of the system. Applying an analogous procedure to the remaining graphs, we obtain a closed system of equations with respect to M_{li}° and M_{ti}° $(i = 1, 2)$. It can be written in the matrix form

$$\left\| \begin{matrix} F_l \\ 0 \\ F_t \\ 0 \end{matrix} \right\| = - \left\| \begin{matrix} M_{l2}^{\circ} \\ M_{l1}^{\circ} \\ M_{t2}^{\circ} \\ M_{t1}^{\circ} \end{matrix} \right\| \left\| \begin{matrix} -1 & V_{ll}^{(2)*} & 0 & V_{lt}^{(2)*} \\ V_{ll}^{(1)*} & -1 & V_{lt}^{(1)*} & 0 \\ 0 & V_{tl}^{(2)*} & -1 & V_{tt}^{(2)*} \\ V_{tl}^{(1)*} & 0 & V_{tt}^{(1)*} & -1 \end{matrix} \right\| \tag{3.1}$$

Expressions for $V_{tl}^{(i)*}$ and $V_{tt}^{(i)*}$ are obtained from $V_{lt}^{(i)*}$ and $V_{ll}^{(i)*}$ by interchanging the subscripts l and t.

Solving this system of linear equations, we obtain

$$M_{l1}^{\circ} = \Delta^{-1}(F_l T_l + F_t S_l), \qquad M_{l2}^{\circ} = \Delta^{-1}(F_l R_l + F_t Q_l)$$
$$T_l = V_{ll}^{(1)*} - V_{tt}^{(2)*} P(1), \qquad S_l = V_{lt}^{(1)*} + V_{lt}^{(2)*} P(1)$$
$$R_l = 1 - V_{tt}^{(1)*} V_{tt}^{(2)*} + V_{lt}^{(1)*} V_{tl}^{(2)*}$$
$$Q_l = V_{ll}^{(2)*} V_{lt}^{(1)*} + V_{lt}^{(2)*} V_{tt}^{(1)*}$$
$$\Delta = 1 - V_{ll}^{(1)*} V_{ll}^{(2)*} - V_{tt}^{(1)*} V_{tt}^{(2)*} - V_{tl}^{(1)*} V_{lt}^{(2)*} - V_{lt}^{(1)*} V_{tl}^{(2)*} + P(1) P(2)$$
$$P(i) = V_{ll}^{(i)*} V_{tt}^{(i)*} - V_{lt}^{(i)*} V_{tl}^{(i)*}$$

The values for M_{t1}° and M_{t2}° are obtained from M_{l1}° and M_{l2}° by replacing the subscript l by t, respectively.

Each graph contains sums of both longitudinal and shear waves; these are, however, easily separated, since at a point (x, y, z) the waves have wave vectors of corresponding type preceding the coordinate in the exponents of the expressions for F_l and F_t.

The total potential $\varphi(\vartheta, k, z, z_0)$ for the plane waves of compression (without taking into account the denominator Δ and the dependence on x and y) can be written in the form

$$\varphi(\vartheta, k, z, z_0) = Z_l(z_0) \exp(b_l z) + Y_l(z_0) \exp(-b_l z) \tag{3.2}$$

in which

$$Z_l(z_0) = [f_0(T_l + R_l) + g_0(S_t + Q_t)] \exp(-b_l z_0)$$
$$Y_l(z_0) = Z_l(z_0) V_{ll}^{(2)} \exp(2b_l h) + V_{lt}^{(2)} [f_0(S_l + Q_l) + g_0(T_t + R_t)] \exp[(b_l + b_t) h - b_t z_0] \tag{3.3}$$

The expression for the total potential $\psi(\vartheta, k, z, z_0)$ for the plane shear waves is obtained by interchanging the subscripts l and t, and also f_0 and g_0, in Eqs. (3.2) and (3.3).

The potentials of the complete sound field for longitudinal and shear waves in a solid layer have the form

$$\varphi = \mathrm{Re}\int_0^\infty e^{ikct}dk \int_{-\pi/2+i\infty}^{\pi/2-i\infty} \frac{1}{\Delta}\varphi(\vartheta, k, z, z_0) H_0^{(1)}(kr\sin\vartheta)\sin\vartheta\, d\vartheta \qquad (3.4)$$

(there is an analogous expression for ψ). Here $H_0^{(1)}(kr\sin\vartheta)$ is a Hankel function of the first kind.

By letting the layer thickness h become infinite, we obtain, in the limit, the problem concerning the reflection of waves from the interface of two media. If, in addition, we assume that the wave vector has a small imaginary part, we can neglect the terms of order $\exp(b_l h)$ and $\exp(b_t h)$ in the resulting expressions. As a result, we obtain

$$\Delta = 1, \qquad Y_l = 0$$
$$Z_l(z_0) = f_0[\exp(-b_l z_0) + V_{ll}^{(1)}\exp(b_l z_0) + g_0 V_{tl}^{(1)}\exp(b_t z_0)$$

and for the potential of the longitudinal waves we obtain the expression given in [1].

4. Normal waves. We consider perturbations associated with the poles of the integrand functions in the expressions (3.4). The poles satisfy the equation $\Delta = 0$. Let $\vartheta = \vartheta_k$ be simple solutions of this equation. Then in the case of a harmonic radiation mode we have

$$\varphi = \mathrm{Re}\sum_k \frac{\sin\vartheta_k \varphi(\vartheta_k, k, z, z_0)}{(d\Delta/d\vartheta)|_{\vartheta=\vartheta_k}} H_0^{(1)}(kr\sin\vartheta_k)$$

(there is an analogous expression for ψ).

Taking the relation (3.2) into account, we can represent each normal wave as a sum of plane waves running in opposite directions along the z-axis. On the lower boundary of the layer the waves of order $\exp(-b_l z)$ and $\exp(-b_t z)$ are incident waves; those of order $\exp(b_l z)$ and $\exp(b_t z)$ are reflected waves. The situation is just the reverse on the upper boundary of the layer. Using the boundary conditions, we obtain the relations for $Z_l(z_0)$, $Z_t(z_0)$, $Y_l(z_0)$ and $Y_t(z_0)$, which were given in [1]. Thus, the results obtained by use of the method of this paper agree with the known results for normal waves in a solid (see [1]).

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